Definitions and key facts for section 1.3

A matrix with only one column is called a **column vector**, or simply a **vector**.

For example,

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 6 \\ -3 \\ 12 \end{bmatrix}, \qquad \text{and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

are all (column) vectors. Here we say $\mathbf{u}$ and $\mathbf{v}$ are elements of $\mathbb{R}^2$, $\mathbf{b}$ is an element of $\mathbb{R}^4$ and $\mathbf{x}$ is an element of $\mathbb{R}^n$.

A key example of a vector is the **zero vector** denoted by $\mathbf{0}$. It is the vector whose entries are all zero. For example,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^2, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^3, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^4, \quad \text{and } \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^n.$$

The algebra of vectors: for any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, we say

- $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$, if their entries are equal;
- the **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector whose entries are the sum of the entries of $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{u} + \mathbf{v}$;
- and the **scalar multiple** of $\mathbf{u}$ by a scalar c (a real number) is the vector $c\mathbf{u}$ obtaining by multiplying each entry of $\mathbf{u}$ by c.

Algebraic properties of (vectors in) $\mathbb{R}^n$
For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^\mathbf{n}$ and all scalars c and d, the following properties hold:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
   where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$
5. $c(\mathbf{u} + \mathbf{v}) = \mathbf{cu} + \mathbf{cv})$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $c(d\mathbf{u}) = (cd)\mathbf{u}$
8. $1\mathbf{u} = \mathbf{u}$

Given vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined

$$\mathbf{y} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_p\mathbf{v_p}$$

is called a **linear combination** of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ with **weights** $c_1, c_2, \ldots, c_p$.

Fact: Vector equations and linear systems
The vector equation

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}$$

has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_n} & \mathbf{b} \end{bmatrix}$$

Notice this implies in particular that $\mathbf{b}$ is a linear combination of $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ exactly when the corresponding system is consistent (and the solution provide the correct weights for the linear combination).

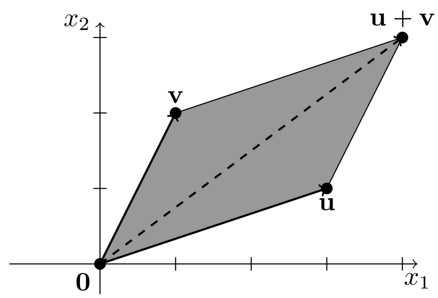
If $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ are in $\mathbb{R}^n$, the set of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ is called the **subset of** $\mathbb{R}^n$ **spanned** (or **generated**) **by** $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$ or more simply, **the span of** $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}$. We denote this set by $\text{Span}\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_p}\}$.

We represent a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$ geometrically in two ways.

- First, as an arrow with initial point at the origin $(0, 0, \ldots, 0)$ and terminal point at $(x_1, x_2, \ldots, x_n)$.
- Secondly, as the point $(x_1, x_2, \ldots, x_n)$.

Parallelogram rule for addition
If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vortex of the parallelogram whose other vertices are $\mathbf{u}, \mathbf{0}$ and $\mathbf{v}$.

For example, in $\mathbb{R}^2$, we sketch the sum of $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



Visualizing scalar multiples
The set of all scalar multiples of a vector $\mathbf{u}$ in $\mathbb{R}^n$ is the collection of vectors on the line through $\mathbf{0}$ and $\mathbf{u}$.

For example, in $\mathbb{R}^2$, we sketch the scalar multiples of $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$: